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Off-shell Bethe vectors and Drinfeld currents

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Abstract

In this paper we compare two constructions of weight functions (off-shell Bethe vectors) for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$. The first construction comes from the algebraic nested Bethe ansatz. The second one is defined in terms of certain projections of products of Drinfeld currents. We show that the two constructions give the same result in tensor products of vector representations of $U_q(\widehat{\mathfrak{gl}}_N)$.

Keywords: Quantum integrable models; Quantum affine algebras

1. Introduction

Off-shell Bethe vectors in integrable models associated with the Lie algebra $\mathfrak{gl}_N$ have appeared in [8] in the framework of the algebraic nested Bethe ansatz. For $N = 2$ they have the form $B(t_1)\cdots B(t_k)v$, where $B(u) = T_{12}(u)$ is an element of the monodromy matrix and v is the highest weight vector of an irreducible finite-dimensional representation of $U_q(\widehat{\mathfrak{gl}}_2)$. For $N > 2$, off-shell Bethe vectors are defined in [8] inductively. They are functions of several complex variables $t^1_1 \dots t^{N-1}_k$ labeled by two indices, the superscript corresponding to a simple root of $\mathfrak{sl}_N$. If the variables $t^1_1 \dots t^{N-1}_k$ satisfy the Bethe ansatz equations, the Bethe vectors are eigenvectors of the transfer matrix of the system.

Off-shell Bethe vectors also serve for integral representations of solutions to the q-difference Knizhnik–Zamolodchikov (qKZ) equations [9–11]. In this case they are known under the name of "weight function". It has been observed in [10] that weight functions have very special comultiplication properties that allow one to express a weight function in a tensor product of representations in terms of weight functions in the tensor factors. The comultiplication properties of weight functions are essential for constructing solutions to the qKZ equations. In this paper we start from these properties and define a weight function as a collection of

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rational functions with values in representations of the quantum affine algebra satisfying suitable comultiplication relations.

A new approach for construction of weight functions has been proposed recently in [7,5]. It is based on the "new realization" of quantum affine algebras [1]. In this approach, the key role is played by certain projections to the intersection of Borel subalgebras of different types of the quantum affine algebra. Those projections were introduced in [6] and were used in [4] to obtain integral formulae for the universal $\mathcal{R}$-matrix of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. It is shown in [7,5] that acting with a projection of a product of Drinfeld currents on highest weight vectors of irreducible finite-dimensional representations of $U_q(\widehat{\mathfrak{gl}}_N)$ one obtains a collection of rational functions with the required comultiplication properties, that is, a weight function.

In this paper we compare two constructions of weight functions for the quantum algebra $U_q(\widehat{\mathfrak{gl}}_N)$. We conjecture that the constructions give the same result for any irreducible finite-dimensional representation of $U_q(\widehat{\mathfrak{gl}}_N)$. We prove this conjecture for tensor products of the vector representations of $U_q(\widehat{\mathfrak{gl}}_N)$. To this end, we show that weight functions defined by the projections and those given by the algebraic Bethe ansatz satisfy the same recurrence relations with respect to the rank N of the algebra. To obtain the recurrence relations we use a generalization of the Ding–Frenkel isomorphisms of two realizations of $U_q(\widehat{\mathfrak{gl}}_N)$.

The paper is organized as follows. In Section 2 we recall two descriptions of $U_q(\widehat{\mathfrak{gl}}_N)$: in terms of the fundamental L-operators and in terms of Drinfeld currents. We define weight functions and symmetric (or modified) weight functions through their coalgebraic properties. We pay special attention to the symmetry properties of weight functions. In Section 3 we describe the construction of a weight function by means of projections of Drinfeld currents. This is done by applying the construction of [5] to the subalgebra $U_q(\widehat{\mathfrak{sl}}_N)$ in $U_q(\widehat{\mathfrak{gl}}_N)$. In Section 4 we describe, following [10], the L-operator construction of the $U_q(\widehat{\mathfrak{gl}}_N)$ weight function. In Section 5 we prove the main result of the paper that the two constructions of weight functions give the same result for tensor products of the vector representations of $U_q(\widehat{\mathfrak{gl}}_N)$. Also, Section 5 contains a description of projections of composed currents, which generalizes the Ding–Frenkel isomorphism.

2. Quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$

2.1. L-operator description

Let $\mathsf{e}_{ij} \in \mathrm{End}(\mathbb{C}^N)$ be a matrix with the only nonzero entry equal to 1 at the intersection of the i-th row and j-th column. Let $R(u,v) \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes \mathbb{C}[[v/u]]$,

$$R(u,v) = \frac{qu - q^{-1}v}{u-v} \sum_{1\le i\le N} \mathsf{e}_{ii}\otimes \mathsf{e}_{ii} + \sum_{1\le i<j\le N} (\mathsf{e}_{ii}\otimes \mathsf{e}_{jj} + \mathsf{e}_{jj}\otimes \mathsf{e}_{ii}) + \frac{q-q^{-1}}{u-v} \sum_{1\le i<j\le N} (v\mathsf{e}_{ij}\otimes \mathsf{e}_{ji} + u\mathsf{e}_{ji}\otimes \mathsf{e}_{ij}), \tag{2.1}$$

be the standard trigonometric R-matrix associated with the vector representation of $\mathfrak{gl}_N$. It satisfies the Yang–Baxter equation

$$R^{12}(u_1,u_2)R^{13}(u_1,u_3)R^{23}(u_2,u_3) = R^{23}(u_2,u_3)R^{13}(u_1,u_3)R^{12}(u_1,u_2), \tag{2.2}$$

and the inversion relation

$$R^{(12)}(u_1,u_2)R^{(21)}(u_2,u_1) = \frac{(qu_1 - q^{-1}u_2)(q^{-1}u_1 - qu_2)}{(u_1-u_2)^2}. \tag{2.3}$$

The algebra $U_q(\widehat{\mathfrak{gl}}_N)$ (with the zero central charge and the gradation operator dropped out) is a unital associative algebra generated by the modes $L^{\pm}_{ij}[\pm k]$, $k \ge 0$, $1 \le i, j \le N$, of the L-operators $L^{\pm}(z) = \sum_{k=0}^{\infty}\sum_{i,j=1}^{N} \mathsf{e}_{ij} \otimes$

$L^{\pm}_{ij}[\pm k]z^{\mp k}$, subject to relations

$$\begin{aligned}
&R(u,v)\cdot(L^{\pm}(u)\otimes\mathbf{1})\cdot(\mathbf{1}\otimes L^{\pm}(v)) = (\mathbf{1}\otimes L^{\pm}(v))\cdot(L^{\pm}(u)\otimes\mathbf{1})\cdot R(u,v)\\
&R(u,v)\cdot(L^{+}(u)\otimes\mathbf{1})\cdot(\mathbf{1}\otimes L^{-}(v)) = (\mathbf{1}\otimes L^{-}(v))\cdot(L^{+}(u)\otimes\mathbf{1})\cdot R(u,v),\\
&L^{+}_{ij}[0] = L^{-}_{ji}[0] = 0, \qquad L^{+}_{kk}[0]L^{-}_{kk}[0] = 1, \quad 1\le i<j\le N, 1\le k\le N.
\end{aligned} \tag{2.4}$$

The coalgebraic structure of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ is defined by the rule

$$\Delta\left(L^{\pm}_{ij}(u)\right) = \sum_{k=1}^{N} L^{\pm}_{kj}(u)\otimes L^{\pm}_{ik}(u). \tag{2.5}$$

2.2. *The current realization of $U_q(\widehat{\mathfrak{gl}}_N)$*

The algebra $U_q(\widehat{\mathfrak{gl}}_N)$ in the current realization (with the zero central charge and the gradation operator dropped out) is generated by the modes of the Cartan currents

$$k^{\pm}_i(z) = \sum_{m\ge 0} k^{\pm}_i[\pm m]z^{\mp m}, \qquad k^{+}_i[0]k^{-}_i[0] = 1,$$

$i = 1,\ldots,N$, and by the modes of the generating functions

$$E_i(z) = \sum_{n\in\mathbb{Z}} E_i[n]z^{-n}, \qquad F_i(z) = \sum_{n\in\mathbb{Z}} F_i[n]z^{-n}, \tag{2.6}$$

$i = 1,\ldots,N-1$, subject to the relations

$$\begin{aligned}
&(q^{-1}z - qw)E_i(z)E_i(w) = E_i(w)E_i(z)(qz - q^{-1}w),\\
&(z-w)E_i(z)E_{i+1}(w) = E_{i+1}(w)E_i(z)(q^{-1}z - qw),\\
&(qz - q^{-1}w)F_i(z)F_i(w) = F_i(w)F_i(z)(q^{-1}z - qw),\\
&(q^{-1}z - qw)F_i(z)F_{i+1}(w) = F_{i+1}(w)F_i(z)(z-w),\\
&k^{\pm}_i(z)F_i(w)\left(k^{\pm}_i(z)\right)^{-1} = \frac{q^{-1}z - qw}{z-w}F_i(w),\\
&k^{\pm}_{i+1}(z)F_i(w)\left(k^{\pm}_{i+1}(z)\right)^{-1} = \frac{qz - q^{-1}w}{z-w}F_i(w),\\
&k^{\pm}_i(z)F_j(w)\left(k^{\pm}_i(z)\right)^{-1} = F_j(w) \quad \text{if } i\ne j, j+1,\\
&k^{\pm}_i(z)E_i(w)\left(k^{\pm}_i(z)\right)^{-1} = \frac{z-w}{q^{-1}z - qw}E_i(w),\\
&k^{\pm}_{i+1}(z)E_i(w)\left(k^{\pm}_{i+1}(z)\right)^{-1} = \frac{z-w}{qz - q^{-1}w}E_i(w),\\
&k^{\pm}_i(z)E_j(w)\left(k^{\pm}_i(z)\right)^{-1} = E_j(w) \quad \text{if } i\ne j, j+1,\\
&[E_i(z),F_j(w)] = \delta_{i,j}\delta(z/w)(q-q^{-1})\left(k^{+}_i(z)/k^{+}_{i+1}(z) - k^{-}_i(w)/k^{-}_{i+1}(w)\right),
\end{aligned} \tag{2.7}$$

together with the Serre relations

$$\begin{aligned}
&\mathrm{Sym}_{z_1,z_2}(E_i(z_1)E_i(z_2)E_{i\pm1}(w) - (q+q^{-1})E_i(z_1)E_{i\pm1}(w)E_i(z_2) + E_{i\pm1}(w)E_i(z_1)E_i(z_2)) = 0,\\
&\mathrm{Sym}_{z_1,z_2}(F_i(z_1)F_i(z_2)F_{i\pm1}(w) - (q+q^{-1})F_i(z_1)F_{i\pm1}(w)F_i(z_2) + F_{i\pm1}(w)F_i(z_1)F_i(z_2)) = 0.
\end{aligned} \tag{2.8}$$

To construct an isomorphism between the L-operator and current realizations of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$, one has to decompose the L-operators into the Gauss coordinates

$$L^{\pm}(z)=\left(\sum_{i=1}^{N}\mathsf{e}_{ii}+\sum_{i<j}^{N}F^{\pm}_{i,j}(z)\mathsf{e}_{ij}\right)\cdot\left(\sum_{i=1}^{N}k^{\pm}_{i}(z)\mathsf{e}_{ii}\right)\cdot\left(\sum_{i=1}^{N}\mathsf{e}_{ii}+\sum_{i<j}^{N}E^{\pm}_{j,i}(z)\mathsf{e}_{ji}\right) \tag{2.9}$$

and for $i=1,\ldots,N-1$ to identify the total currents and the linear combinations of the nearest to the diagonal Gauss coordinates [2]

$$E_i(z)=E^{+}_{i+1,i}(z)-E^{-}_{i+1,i}(z),\qquad F_i(z)=F^{+}_{i,i+1}(z)-F^{-}_{i,i+1}(z). \tag{2.10}$$

The diagonal Gauss coordinates of the L-operators coincide with the Cartan currents $k^{\pm}_i(z)$ and are denoted by the same letter. The results of [2] say nothing about relations between Gauss coordinates $F^{\pm}_{i,j}(z)$ and $E^{\pm}_{j,i}(z)$ of the L-operators for $j-i>1$ and the currents $F_i(z)$, $E_i(z)$. Some of these relations, that we will need for our construction, will be described in Section 5.2.

The *current* Hopf structure for the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ has been defined in [1],

$$\begin{aligned}
&\Delta^{(D)}\left(E_i(z)\right)=E_i(z)\otimes 1+k^{-}_i(z)\left(k^{-}_{i+1}(z)\right)^{-1}\otimes E_i(z),\\
&\Delta^{(D)}\left(F_i(z)\right)=1\otimes F_i(z)+F_i(z)\otimes k^{+}_i(z)\left(k^{+}_{i+1}(z)\right)^{-1},\\
&\Delta^{(D)}\left(k^{\pm}_i(z)\right)=k^{\pm}_i(z)\otimes k^{\pm}_i(z).
\end{aligned} \tag{2.11}$$

We consider two types of Borel subalgebras of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$. Borel subalgebras $U_q(\mathfrak{b}_{\pm})\subset U_q(\widehat{\mathfrak{gl}}_N)$ are generated by the modes of the L-operators $L^{\pm}(z)$, respectively.

Another type of Borel subalgebras is related to the current realization of $U_q(\widehat{\mathfrak{gl}}_N)$. The Borel subalgebra $U_F\subset U_q(\widehat{\mathfrak{gl}}_N)$ is generated by modes of the currents $F_i[n]$, $k^{+}_j[m]$, $i=1,\ldots,N-1$, $j=1,\ldots,N$, $n\in\mathbb{Z}$ and $m\geq 0$. The Borel subalgebra $U_E\subset U_q(\widehat{\mathfrak{gl}}_N)$ is generated by modes of the currents $E_i[n]$, $k^{-}_j[-m]$, $i=1,\ldots,N-1$, $j=1,\ldots,N$, $n\in\mathbb{Z}$ and $m\geq 0$. We will consider also a subalgebra $U'_F\subset U_F$, generated by the elements $F_i[n]$, $k^{+}_j[m]$, $i=1,\ldots,N-1$, $j=1,\ldots,N$, $n\in\mathbb{Z}$ and $m>0$, and a subalgebra $U'_E\subset U_E$ generated by the elements $E_i[n]$, $k^{-}_j[-m]$, $i=1,\ldots,N-1$, $j=1,\ldots,N$, $n\in\mathbb{Z}$ and $m>0$. Further, we will be interested in the intersections,

$$U^{-}_f=U'_F\cap U_q(\mathfrak{b}_-),\qquad U^{+}_F=U_F\cap U_q(\mathfrak{b}_+) \tag{2.12}$$

and will describe properties of projections to these intersections.

2.3. A weight function

We call a vector v *a weight singular vector* if it is annihilated by any non-negative mode of the currents $E_i[n]$, $i=1,\ldots,N-1$, $n\geq 0$ and is an eigenvector for the Cartan currents $k^{+}_i(z)$, $i=1,\ldots,N$,

$$E^{+}_{i+1,i}(z)\cdot v=0,\qquad k^{+}_i(z)\cdot v=\Lambda_i(z)\,v, \tag{2.13}$$

where $\Lambda_i(z)$ is a meromorphic function, decomposed as a power series in z^{-1}. The L-operator (2.9), acting on a weight singular vector v, becomes upper-triangular:

$$L^{+}_{ij}(z)v=0,\quad i>j,\qquad L^{+}_{ii}(z)v=\Lambda_i(z)v,\quad i=1,\ldots,N. \tag{2.14}$$

We define a weight function by its comultiplication properties.

Let Π be the set $\{1,\ldots,N-1\}$ of indices of simple positive roots of $\mathfrak{gl}_N$. A finite collection $I=\{i_1,\ldots,i_n\}$ with a linear ordering $i_i\prec\cdots\prec i_n$ and a map $\iota: I\to\Pi$ is called an *ordered Π-multiset*. Sometimes, we denote the map ι by ι_I and call it a "*colouring map*". A morphism between two ordered Π-multisets I and J is a map $m: I\to J$ that respects the orderings in I and J and intertwines the colouring maps: $\iota_J m=m\iota_I$. In particular, any subset $I'\subset I$ of a Π-ordered multiset has a unique structure of Π-ordered multiset, such that the inclusion map is a morphism of Π-ordered multisets.

To each Π-ordered multiset $I=\{i_1,\ldots,i_n\}$ we attach an ordered set of variables $\{t_i\mid i\in I\}=\{t_{i_1},\ldots,t_{i_n}\}$. Each variable has its own "colour": $\iota(i_k)\in\Pi$.

Let i and j be elements of some ordered Π-multiset. Define a rational function

$$\gamma(t_i,t_j)=\begin{cases}\dfrac{qt_i-q^{-1}t_j}{t_i-t_j}, & \text{if } \iota(i)=\iota(j)+1,\\ \dfrac{t_i-t_j}{q^{-1}t_i-qt_j}, & \text{if } \iota(j)=\iota(i)+1,\\ \dfrac{q^{-1}t_i-qt_j}{qt_i-q^{-1}t_j}, & \text{if } \iota(i)=\iota(j),\\ 1, & \text{otherwise.}\end{cases} \tag{2.15}$$

Assume that for any representation V of $U_q(\widehat{\mathfrak{gl}}_N)$, generated by a weight singular vector v, and any ordered Π-multiset $I=\{i_1,\dots i_n\}$, there is a V-valued rational function $w_{V,I}(t_{i_1},\dots,t_{i_n})\in V$ depending on the variables $\{t_i \mid i\in I\}$. We call such a collection of rational functions a *weight function* w, if:

(a) The rational function, corresponding to the empty set, is equal to v,

$$w_{V,\emptyset}\equiv v. \tag{2.16}$$

(b) The function $w_{V,I}(t_{i_1},\dots,t_{i_n})$ depends only on an isomorphism class of an ordered Π-multiset, that is, for any isomorphism $f\colon I\to J$ of ordered Π-multisets we have

$$w_{V,I}(t_{f(i)}|_{i\in I})=w_{V,J}(t_j|_{j\in J}). \tag{2.17}$$

(c) The functions $w_{V,I}$ satisfy the following comultiplication property. Let $V=V_1\otimes V_2$ be a tensor product of two representations generated by the singular vectors v_1, v_2 and weight series $\{\Lambda_b^{(1)}(u)\}$ and $\{\Lambda_b^{(2)}(u)\}$, $b=1,\dots,N$. Then for any multiset I we have

$$w_{V,I}(t_i|_{i\in I})=\sum_{I=I_1\coprod I_2} w_{V_1,I_1}(t_i|_{i\in I_1})\otimes w_{V_2,I_2}(t_i|_{i\in I_2})\,\Phi_{I_1,I_2}(t_i|_{i\in I})\prod_{j\in I_1}\frac{\Lambda^{(2)}_{\iota(j)}(t_j)}{\Lambda^{(2)}_{\iota(j)+1}(t_j)}, \tag{2.18}$$

where

$$\Phi_{I_1,I_2}(t_i|_{i\in I})=\prod_{\substack{i\in I_1,j\in I_2\\ i\prec j}}\gamma(t_i,t_j). \tag{2.19}$$

The summation in (2.18) runs over all possible decompositions of the ordered multiset I into a disjoint union of two non-intersecting ordered submultisets I_1 and I_2.

Note that the comultiplication property relation (2.18) is not a recurrence relation, that is, it does not allow us to reconstruct functions $w_{V,I}$ for all ordered multisets I starting from the functions which correspond to the multisets with $|I|=1$.

Let $I=\{i_1,\dots,i_n\}$ and $J=\{j_1,\dots,j_n\}$ be two ordered Π-multisets. Let $\sigma: I\to J$ be an invertible map, which intertwines the colouring maps, $\iota_J\sigma=\sigma\iota_I$, but does not necessarily respect the orderings in I and J (that is, σ is a "permutation" on classes of isomorphisms of ordered multisets).

Let $w(t_j|_{j\in J})$ be a function of the variables $t_j|_{j\in J}$. Define a *pullback* ${}^{\sigma,\gamma}w(t_i|_{i\in I})$ by the rule

$${}^{\sigma,\gamma}w(t_i|_{i\in I})=w(t_{\sigma(i)}|_{i\in I})\prod_{\substack{i,j\in I\\ i\prec j,\sigma(j)\prec\sigma(i)}}\gamma(t_i,t_j). \tag{2.20}$$

Let I and I' be ordered Π-multisets, and $\sigma: I\to I'$ an invertible map, intertwining the colouring maps. Then its restriction to any subset $J\subset I$ is an invertible map of J to $\sigma(J)$, intertwining the colouring maps.

Proposition 2.1. *Let w be a weight function, I, J ordered Π-multisets, and $\sigma\colon I\to J$ an invertible map, intertwining the colouring maps. Then we have*

$${}^{\sigma,\gamma}w_{V,J}(t_i|_{i\in I})=\sum_{I=I_1\coprod I_2}{}^{\sigma,\gamma}w_{V_1,\sigma(I_1)}(t_i|_{i\in I_1})\otimes{}^{\sigma,\gamma}w_{V_2,\sigma(I_2)}(t_i|_{i\in I_2})\,\Phi_{I_1,I_2}(t_i|_{i\in I})\prod_{j\in I_1}\frac{\Lambda^{(2)}_{\iota(j)}(t_j)}{\Lambda^{(2)}_{\iota(j)+1}(t_j)}.$$

The proposition means that the pullback operation (2.20) is compatible with the comultiplication rule (2.18).

We call a weight function w *q-symmetric* if for any ordered Π-multisets I and J and an invertible map $\sigma: I \to J$, intertwining the colouring maps, we have

$$^{\sigma,\gamma}w_{V,J}(t_i|_{i\in I}) = w_{V,I}(t_i|_{i\in I}).$$

2.4. A modified weight function

Given elements i, j of some ordered multiset define two functions $\tilde{\gamma}(t_i,t_j)$ and $\beta(t_i,t_j)$ by the formulae

$$\tilde{\gamma}(t_i,t_j) = \begin{cases} \dfrac{t_i-t_j}{qt_i-q^{-1}t_j}, & \text{if } \iota(i)=\iota(j)+1,\\ \dfrac{q^{-1}t_i-qt_j}{t_i-t_j}, & \text{if } \iota(j)=\iota(i)+1,\\ 1, & \text{otherwise} \end{cases}$$

and

$$\beta(t_i,t_j) = \begin{cases} \dfrac{q^{-1}t_i-qt_j}{t_i-t_j}, & \text{if } \iota(i)=\iota(j),\\ 1, & \text{otherwise.} \end{cases} \tag{2.21}$$

A collection of rational V-valued functions $\mathbf{w}_{V,I}(t_i|_{i\in I})$, depending on a representation V of $U_q(\widehat{\mathfrak{gl}}_N)$, generated by a weight singular vector v, and an ordered Π-multiset I, is called a *modified weight function* $\mathbf{w}$ if it satisfies conditions (a), (b); see (2.16) and (2.17), and condition (c′):

(c′) Let $V = V_1\otimes V_2$ be a tensor product of two representations generated by the singular vectors v_1, v_2 and weight series $\{\Lambda_b^{(1)}(u)\}$ and $\{\Lambda_b^{(2)}(u)\}$, $b=1,\dots,N$. Then for any multiset I we have

$$\mathbf{w}_{V,I}(t_i|_{i\in I}) = \sum_{I=I_1\coprod I_2} \mathbf{w}_{V_1,I_1}(t_i|_{i\in I_1})\otimes \mathbf{w}_{V_2,I_2}(t_i|_{i\in I_2})\cdot \tilde{\Phi}_{I_1,I_2}(t_i|_{i\in I})\prod_{j\in I_1}\Lambda^{(2)}_{\iota(j)}(t_j)\prod_{j\in I_2}\Lambda^{(1)}_{\iota(j)+1}(t_j), \tag{2.22}$$

where

$$\tilde{\Phi}_{I_1,I_2}(t_i|_{i\in I}) = \prod_{i\in I_1, j\in I_2}\beta(t_i,t_j)\prod_{\substack{i\in I_2, j\in I_1\\ i\prec j}}\tilde{\gamma}(t_i,t_j).$$

Let $I=\{i_1,\dots,i_n\}$ and $J=\{j_1,\dots,j_n\}$ be two ordered Π-multisets, and $\sigma: I\to J$ an invertible map, intertwining the colouring maps, $\iota_J\sigma = \sigma\iota_I$. Let $\mathbf{w}(t_j|_{j\in J})$ be a function of the variables $t_j|_{j\in J}$. Define a pullback $^{\sigma,\tilde{\gamma}}\mathbf{w}(t_i|_{i\in I})$ by the rule

$$^{\sigma,\tilde{\gamma}}\mathbf{w}(t_i|_{i\in I}) = \mathbf{w}(t_{\sigma(i)}|_{i\in I})\prod_{\substack{i,j\in I\\ i\prec j,\sigma(j)\prec\sigma(i)}}\tilde{\gamma}(t_i,t_j). \tag{2.23}$$

Proposition 2.2. *Let* $\mathbf{w}$ *be a modified weight function,* I, J *ordered* Π*-multisets, and* $\sigma: I\to J$ *an invertible map, intertwining the colouring maps. Then we have*

$$\begin{aligned} ^{\sigma,\tilde{\gamma}}\mathbf{w}_{V,J}(t_i|_{i\in I}) = &\sum_{I=I_1\coprod I_2} {}^{\sigma,\tilde{\gamma}}\mathbf{w}_{V_1,\sigma(I_1)}(t_i|_{i\in I_1})\otimes {}^{\sigma,\tilde{\gamma}}\mathbf{w}_{V_2,\sigma(I_2)}(t_i|_{i\in I_2})\\ &\times\tilde{\Phi}_{I_1,I_2}(t_i|_{i\in I})\prod_{j\in I_1}\Lambda^{(2)}_{\iota(j)}(t_j)\prod_{j\in I_2}\Lambda^{(1)}_{\iota(j)+1}(t_j). \end{aligned}$$

We call a modified weight function $\mathbf{w}$ *q-symmetric* if for any two ordered Π-multisets I and J and an invertible map $\sigma: I \to J$, intertwining the colouring maps, we have

$$^{\sigma,\tilde{\gamma}}\mathbf{w}_{V,J}(t_i|_{i\in I}) = \mathbf{w}_{V,I}(t_i|_{i\in I}).$$

For an ordered Π-multiset $I = \{i_1, i_2, \dots, i_n\}$, let $\bar{I} = \{i_n, i_{n-1}, \dots, i_1\}$ be the ordered Π-multiset with the colouring map, $\iota_{\bar{I}}(i_k) = \iota_I(i_k)$, $k = 1, \dots, n$.

Proposition 2.3. (i) *Let w be a weight function. Then the collection $\mathbf{w}_{V,I}(t_i|_{i\in I})$, where*

$$\mathbf{w}_{V,I}(t_i|_{i\in I}) = w_{V,\bar{I}}(t_i|_{i\in\bar{I}}) \prod_{i\prec j} \beta(t_i, t_j) \prod_{i\in I} \Lambda_{\iota(i)+1}(t_i) \tag{2.24}$$

is a modified weight function.

(ii) *Let $\mathbf{w}$ be a modified weight function. Then the collection $w_{V,I}(t_i|_{i\in I})$ where*

$$w_{V,I}(t_i|_{i\in I}) = \mathbf{w}_{V,\bar{I}}(t_i|_{i\in\bar{I}}) \prod_{i\prec j} \frac{1}{\beta(t_j, t_i)} \prod_{i\in I} \frac{1}{\Lambda_{\iota(i)+1}(t_i)}$$

is a weight function.

(iii) *If w is a q-symmetric weight function, then $\mathbf{w}$ is q-symmetric modified weight function, and vice versa.*

The last proposition means that we have a bijection between weight functions and modified weight functions.

3. Weight functions and Drinfeld currents

3.1. Quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_N)$

We are using two descriptions of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_N)$: in terms of Chevalley generators and the current realization.

The algebra $U_q(\widehat{\mathfrak{sl}}_N)$ (with zero central charge and the grading element dropped out) is generated by the Chevalley generators $e_{\pm\alpha_i}$, $k_{\alpha_i}^{\pm 1}$, where $i = 0, 1, \dots, N-1$ and $\prod_{i=0}^{N} k_{\alpha_i} = 1$, subject to relations

$$k_{\alpha_i} e_{\pm\alpha_j} k_i^{-1} = q_i^{\pm a_{ij}} e_{\pm\alpha_j}, \qquad [e_i, e_{-\alpha_j}] = \delta_{ij} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q_i - q_i^{-1}}, \tag{3.1}$$

$$\sum_{r=0}^{m_{i,j}} (-1)^r e_{\pm\alpha_i}^{(r)} e_{\pm\alpha_j} e_{\pm\alpha_i}^{(m_{i,j}-r)} = 0, \quad \text{where } m_{i,j} = 1 - (\alpha_i, \alpha_j), i \neq j,$$

$$e_{\pm\alpha_i}^{(r)} = \frac{e_{\pm\alpha_i}^r}{[k]_q!}, \quad [k]_q! = [k]_q[k-1]_q \cdots [2]_q[1]_q, \quad [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}, \tag{3.2}$$

and $a_{i,j} = (\alpha_i, \alpha_j)$ is the Cartan matrix of the affine algebra $\widehat{\mathfrak{sl}}$.

The comultiplication map is given by the formulae

$$\begin{aligned} \Delta(e_{\alpha_i}) &= e_{\alpha_i} \otimes 1 + k_{\alpha_i} \otimes e_{\alpha_i}, \\ \Delta(e_{-\alpha_i}) &= 1 \otimes e_{-\alpha_i} + e_{-\alpha_i} \otimes k_{\alpha_i}^{-1}, \\ \Delta(k_{\alpha_i}) &= k_{\alpha_i} \otimes k_i. \end{aligned} \tag{3.3}$$

In the current realization, $U_q(\widehat{\mathfrak{sl}}_N)$ is generated by the elements $e_i[n]$, $f_i[n]$, where $i = 1, \dots, N-1$, $n \in \mathbb{Z}$; $\psi_i^{\pm}[n]$, $i = 1, \dots, N-1$, $n \geq 0$, $\psi_i^{-}[0] = \left(\psi_i^{+}[0]\right)^{-1}$. They are combined into generating functions

$$e_i(z) = \sum_{n\in\mathbb{Z}} e_i[n] z^{-n}, \qquad f_i(z) = \sum_{n\in\mathbb{Z}} f_i[n] z^{-n}, \qquad \psi_i^{\pm}(z) = \sum_{n\geq 0} \psi_i^{\pm}[n] z^{\mp n},$$

which satisfy the following relations:

$$(z-q^{(\alpha_i,\alpha_j)}w)e_i(z)e_j(w)=e_j(w)e_i(z)(q^{(\alpha_i,\alpha_j)}z-w),$$
$$(z-q^{-(\alpha_i,\alpha_j)}w)f_i(z)f_j(w)=f_j(w)f_i(z)(q^{-(\alpha_i,\alpha_j)}z-w),$$
$$\psi_i^{\pm}(z)e_j(w)\left(\psi_i^{\pm}(z)\right)^{-1}=\frac{(q^{(\alpha_i,\alpha_j)}z-w)}{(z-q^{(\alpha_i,\alpha_j)}w)}e_j(w),$$
$$\psi_i^{\pm}(z)f_j(w)\left(\psi_i^{\pm}(z)\right)^{-1}=\frac{(q^{-(\alpha_i,\alpha_j)}z-w)}{(z-q^{-(\alpha_i,\alpha_j)}w)}f_j(w),$$
$$\psi_i^{\mu}(z)\psi_j^{\nu}(w)=\psi_j^{\nu}(w)\psi_i^{\mu}(z),\quad \mu,\nu=\pm,$$
$$[e_i(z),f_j(w)]=\frac{\delta_{ij}\delta(z/w)}{q-q^{-1}}\left(\psi_i^{+}(z)-\psi_i^{-}(w)\right),$$

and

$$\operatorname*{Sym}_{z_1,z_2}\Big(e_i(z_1)e_i(z_2)e_j(w)-(q+q^{-1})e_i(z_1)e_j(w)e_i(z_2)+e_j(w)e_i(z_1)e_i(z_2)\Big)=0,$$
$$\operatorname*{Sym}_{z_1,z_2}\Big(f_i(z_1)f_i(z_2)f_j(w)-(q+q^{-1})f_i(z_1)f_j(w)f_i(z_2)+f_j(w)f_i(z_1)f_i(z_2)\Big)=0,$$

where $i-j=\pm1$.

The two realizations are related by the formulae

$$k_{\alpha_i}=\psi_i^{+}[0],\qquad e_{\alpha_i}=e_i[0],\qquad e_{-\alpha_i}=f_i[0],\quad i=1,\dots,N-1,$$
$$e_{\alpha_0}=[e_1[0],[e_2[0],\dots,[e_{N-2}[0],e_{N-1}[-1]]_q\dots]_q,$$
$$e_{-\alpha_0}=[\dots[f_{N-1}[1],f_{N-2}[0]]_{q^{-1}},\dots,f_2[0]]_{q^{-1}},f_1[0]]_{q^{-1}},$$

where $[e_i[k],e_j[l]]_q=e_i[k]e_j[l]-q^{(\alpha_i,\alpha_j)}e_j[l]e_i[k]$ and $[f_i[k],f_j[l]]_{q^{-1}}=f_j[l]f_i[k]-q^{-(\alpha_i,\alpha_j)}f_i[k]f_j[l]$.

The Drinfeld comultiplication $\Delta^{(D)}$ for the algebra $U_q(\widehat{\mathfrak{sl}}_N)$ looks as follows:

$$\Delta^{(D)}e_i(z)=e_i(z)\otimes 1+\psi_i^{-}(z)\otimes e_i(z),$$
$$\Delta^{(D)}f_i(z)=1\otimes f_i(z)+f_i(z)\otimes\psi_i^{+}(z),$$
$$\Delta^{(D)}\psi_i^{\pm}(z)=\psi_i^{\pm}(z)\otimes\psi_i^{\pm}(z).$$

The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_N)$ has two types of Borel subalgebras. The Borel subalgebras $U_q(\mathfrak{b}_{\pm}^{\mathfrak{sl}})\subset U_q(\widehat{\mathfrak{sl}}_N)$ are generated by the Chevalley generators $e_{\alpha_i},k_{\alpha_i}^{\pm1}$, $i=0,\dots,N-1$ and $e_{-\alpha_i},k_{\alpha_i}^{\pm1}$, $i=0,\dots,N-1$, respectively. They contain Hopf coideals $U_q(\mathfrak{n}_{\pm}^{\mathfrak{sl}})\subset U_q(\mathfrak{b}_{\pm}^{\mathfrak{sl}})$, generated by the Chevalley generators e_{α_i}, and $e_{-\alpha_i}$, $i=0,\dots,N-1$, respectively.

The Borel subalgebra $U_F^{\mathfrak{sl}}\subset U_q(\widehat{\mathfrak{sl}}_N)$ is generated by the elements $f_i[n]$, where $i=1,\dots,N-1$, $n\in\mathbb{Z}$ and $\psi_i^{+}[n]$, $i=1,\dots,N-1$, $n\geq 0$. The Borel subalgebra $U_E^{\mathfrak{sl}}\subset U_q(\widehat{\mathfrak{sl}}_N)$ is generated by the elements $e_i[n]$, where $i=1,\dots,N-1$, $n\in\mathbb{Z}$ and $\psi_i^{-}[n]$, $i=1,\dots,N-1$, $n\geq 0$. We are interested in their intersections,

$$U_f^{\mathfrak{sl}-}=U_F^{\mathfrak{sl}}\cap U_q(\mathfrak{n}_-^{\mathfrak{sl}}),\qquad U_F^{\mathfrak{sl}+}=U_F^{\mathfrak{sl}}\cap U_q(\mathfrak{b}_+^{\mathfrak{sl}}).\tag{3.4}$$

According to [5], these intersections satisfy coideal properties

$$\Delta^{(D)}(U_F^{\mathfrak{sl}+})\subset U_q(\widehat{\mathfrak{sl}}_N)\otimes U_F^{\mathfrak{sl}+},\qquad \Delta^{(D)}(U_f^{\mathfrak{sl}-})\subset U_f^{\mathfrak{sl}-}\otimes U_q(\widehat{\mathfrak{sl}}_N)$$

and the multiplication m in $U_q(\widehat{\mathfrak{sl}}_N)$ induces an isomorphism of vector spaces

$$m: U_f^{\mathfrak{sl}-}\otimes U_F^{\mathfrak{sl}+}\to U_F^{\mathfrak{sl}}.$$

The projection operator $P: U_F^{\mathfrak{sl}}\to U_F^{\mathfrak{sl}+}$ is defined by the rule

$$P(f_-f_+)=\varepsilon(f_-)f_+,\qquad f_-\in U_f^{\mathfrak{sl}-}\subset U_q(\widehat{\mathfrak{sl}}_N),\ f_+\in U_F^{\mathfrak{sl}+}\subset U_q(\widehat{\mathfrak{sl}}_N).\tag{3.5}$$

3.2. The embedding of $U_q(\widehat{\mathfrak{sl}}_N)$ to $U_q(\widehat{\mathfrak{gl}}_N)$

Consider the embedding $\Theta : U_q(\widehat{\mathfrak{sl}}_N) \hookrightarrow U_q(\widehat{\mathfrak{gl}}_N)$ given by the following formulae in the current realizations of $U_q(\widehat{\mathfrak{sl}}_N)$ and $U_q(\widehat{\mathfrak{gl}}_N)$:

$$\begin{aligned} &\Theta(e_i(z)) = (q-q^{-1})^{-1}E_i(q^{-i+1}z), \qquad \Theta(f_i(z)) = (q-q^{-1})^{-1}F_i(q^{-i+1}z),\\ &\Theta(\psi_i^{\pm}(z)) = k_i^{\pm}(q^{-i+1}z)\left(k_{i+1}^{\pm}(q^{-i+1}z)\right)^{-1}. \end{aligned} \tag{3.6}$$

Let $g^+(z)$, $g^-(z)$ be power series with coefficients in $\mathbb{C}$,

$$\begin{aligned} &g^+(z) = g_0^+ + g_1z^{-1} + \cdots + g_nz^{-n} + \cdots,\\ &g^-(z) = g_0^- + g_{-1}z + \cdots + g_{-n}z^n + \cdots, \end{aligned} \tag{3.7}$$

satisfying the condition

$$g_0^+g_0^- = 1. \tag{3.8}$$

A pair $g^{\pm}(z)$ defines an automorphism $T_{g^+(z),\,g^-(z)}$ of the algebra $U_q(\widehat{\mathfrak{gl}}_N)$ by the rule

$$T_{g^+(z),\,g^-(z)}L^+(z) = g^+(z)L^+(z), \qquad T_{g^+(z),\,g^-(z)}L^-(z) = g^-(z)L^-(z). \tag{3.9}$$

The following facts are well known.

Proposition 3.1. (i) *The embedding Θ is a morphism of Hopf algebras with respect to any of the comultiplications Δ or $\Delta^{(D)}$.*

(ii) *The image of Θ is the subalgebra of invariants of all automorphisms $T_{g^+(z),g^-(z)}$ in $U_q(\widehat{\mathfrak{gl}}_N)$.*

(iii) *The embedding Θ maps the Borel subalgebras $U_q(\mathfrak{b}_{\pm}^{\mathfrak{sl}}) \subset U_q(\widehat{\mathfrak{sl}}_N)$ into the corresponding Borel subalgebras $U_q(\mathfrak{b}_{\pm}) \subset U_q(\widehat{\mathfrak{gl}}_N)$, and the current Borel subalgebra $U_F^{\mathfrak{sl}} \subset U_q(\widehat{\mathfrak{sl}}_N)$ into the current Borel subalgebra $U_F \subset U_q(\widehat{\mathfrak{gl}}_N)$.*

3.3. Projections

Clearly, the Borel subalgebras in $U_q(\widehat{\mathfrak{sl}}_N)$ and $U_q(\widehat{\mathfrak{gl}}_N)$ differ only by Cartan currents. We have N Cartan currents $k_i^+(z)$, $i = 1,\ldots,N$, in $U_q(\mathfrak{b}_+) \subset U_q(\widehat{\mathfrak{gl}}_N)$ and in $U_F \subset U_q(\widehat{\mathfrak{gl}}_N)$, while we have $N-1$ Cartan currents $\psi_i^+(z)$, $i = 1,\ldots,N-1$, in $U_q(\mathfrak{b}_+^{\mathfrak{sl}}) \subset U_q(\widehat{\mathfrak{sl}}_N)$ and in $U_F^{\mathfrak{sl}} \subset U_q(\widehat{\mathfrak{sl}}_N)$, and $\Theta(\psi_i^+(z)) = k_i^+(q^{-i+1}z)\left(k_{i+1}^+(q^{-i+1}z)\right)^{-1}$.

We can choose modes of one of the currents $k_1^+(z)$ as generators of an abelian subalgebra A_1. Then the multiplication in $U_q(\widehat{\mathfrak{gl}}_N)$ establishes an isomorphism (of vector spaces) between the Borel subalgebra $U_q(\mathfrak{b}_+) \subset U_q(\widehat{\mathfrak{gl}}_N)$ and the tensor product of A_1 and the image of Borel subalgebra $U_q(\mathfrak{b}_+^{\mathfrak{sl}}) \subset U_q(\widehat{\mathfrak{sl}}_N)$. An analogous statement holds for the current Borel subalgebras $U_F^{\mathfrak{sl}} \subset U_q(\widehat{\mathfrak{sl}}_N)$ and $U_F \subset U_q(\widehat{\mathfrak{gl}}_N)$.

This observation implies that the algebras $U_f^- = U_F' \cap U_q(\mathfrak{b}_-) \subset U_q(\widehat{\mathfrak{gl}}_N)$ and $U_F^+ = U_F \cap U_q(\mathfrak{b}_+) \subset U_q(\widehat{\mathfrak{gl}}_N)$, see (2.12), satisfy the same properties as the analogous subalgebras (3.4) of $U_q(\widehat{\mathfrak{sl}}_N)$. Namely, they are coideals,

$$\Delta^{(D)}(U_F^+) \subset U_q(\widehat{\mathfrak{gl}}_N) \otimes U_F^+, \qquad \Delta^{(D)}(U_f^-) \subset U_f^- \otimes U_q(\widehat{\mathfrak{gl}}_N),$$

the multiplication m in $U_q(\widehat{\mathfrak{gl}}_N)$ induces an isomorphism of vector spaces

$$m\colon U_f^- \otimes U_F^+ \to U_F,$$

and the projection operator $P\colon U_F \subset U_q(\widehat{\mathfrak{gl}}_N) \to U_F^+$ is defined similar to (3.5):

$$P(f_-f_+) = \varepsilon(f_-)f_+, \quad f_- \in U_f^- \subset U_q(\widehat{\mathfrak{gl}}_N),\ f_+ \in U_F^+ \subset U_q(\widehat{\mathfrak{gl}}_N). \tag{3.10}$$

Proposition 3.1 yields that the definitions (3.5) and (3.10) are consistent, that is, for any element $f \in U_F \subset U_q(\widehat{\mathfrak{sl}}_N)$, we have

$$\Theta\left(P(f)\right) = P\left(\Theta(f)\right), \tag{3.11}$$

where P in the left hand side is the projection operator (3.5) and P in the right hand side is the projection operator (3.10).

3.4. A construction of the weight function

Let V be a representation of $U_q(\widehat{\mathfrak{gl}}_N)$ generated by a singular vector v. Let $I = \{i_1, \dots, i_n\}$ be an ordered Π-multiset. Set

$$w_{V,I}(\{t_i|_{i\in I}\}) = P\left(F_{\iota(i_1)}(t_{i_1}) \cdots F_{\iota(i_n)}(t_{i_n})\right) v. \tag{3.12}$$

Theorem 1. *A collection of V-valued rational functions $w_{V,I}(\{t_i|_{i\in I}\})$, given by (3.12), is a q-symmetric weight function.*

Proof. Due to (3.11) and (3.6),

$$w_{V,I}(\{t_i|_{i\in I}\}) = (q - q^{-1})^n \Theta\left(P\left(f_{\iota(i_1)}(\tilde{t}_{i_1}) \cdots f_{\iota(i_n)}(\tilde{t}_{i_n})\right)\right) v,$$

where $\tilde{t}_{i_1}, \dots, \tilde{t}_{i_n}$ are the variables $t_{i_1}, \dots, t_{i_n}$, shifted by some powers of q. Therefore, the collection of functions $w_{V,I}(\{t_i|_{i\in I}\})$ is a $U_q(\widehat{\mathfrak{sl}}_N)$ weight function up to a certain shift of variables, and Theorem 1 is a particular case of Theorem 4 in [5]. Let us recall that the key assertion used in the Proof of Theorem 4 in [5] is the following relation for the comultiplications Δ, $\Delta^{(D)}$, and the projection operator P, that holds in $U_q(\widehat{\mathfrak{g}})$ for any simple Lie algebra $\mathfrak{g}$: for any element $f \in U_F^{\mathfrak{g}}$ and any singular vectors v_1, v_2, one has

$$\Delta(P(f))\, v_1 \otimes v_2 = (P \otimes P)\Delta^{(D)}(f)\, v_1 \otimes v_2. \tag{3.13}$$

The q-symmetry of the weight function w follows from the defining relations (2.7). $\square$

For an ordered Π-multiset $I = \{i_1, \dots, i_n\}$ set

$$\mathbf{w}^{\mathrm{P}}_{V,I}(\{t_i|_{i\in I}\}) = P\left(F_{\iota(i_n)}(t_{i_n}) \cdots F_{\iota(i_1)}(t_{i_1})\right) v \cdot \prod_{i \prec j} \beta(t_i, t_j) \prod_{i \in I} \Lambda_{\iota(i)+1}(t_i), \tag{3.14}$$

where $\beta(t_i, t_j)$ is defined by (2.21). Theorem 1 and Proposition 2.3 imply the following statement.

Corollary 3.2. *A collection of V-valued rational functions $\mathbf{w}^{\mathrm{P}}_{V,I}(\{t_i|_{i\in I}\})$, given by (3.14), is a q-symmetric modified weight function.*

4. L-operators and modified weight functions

4.1. $U_q(\widehat{\mathfrak{gl}}_N)$ monodromy

We borrow the construction below from [10]. We will need only one L-operator, say $L^+(z)$, which we denote as $L(z)$. It generates the Borel subalgebra $U_q(\mathfrak{b}_+)$; see Section 2.1.

Let M be a non-negative integer. Let $L^{(k)}(z) \in \left(\mathbb{C}^N\right)^{\otimes M}$ be the L-operator acting as $L(z)$ on k-th tensor factor in the product $\left(\mathbb{C}^N\right)^{\otimes M}$ and as the identity operator in all other factors. Consider a series in M variables

$$\mathbb{T}_{[M]}(u_1, \dots, u_M) = L^{(1)}(u_1) \cdots L^{(M)}(u_M) \cdot \mathbb{R}^{(M,\dots,1)}(u_M, \dots, u_1) \tag{4.1}$$

with coefficients in $\left(\mathrm{End}(\mathbb{C}^N)\right)^{\otimes M} \otimes U_q(\mathfrak{b}_+)$, where

$$\mathbb{R}^{(M,\dots,1)}(u_M, \dots, u_1) = \prod_{1 \le i < j \le M}^{\longleftarrow} R^{(ji)}(u_j, u_i). \tag{4.2}$$

In the ordered product of R-matrices (4.2) the factor $R^{(ji)}$ is to the left of the factor $R^{(ml)}$ if $j > m$, or $j = m$ and $i > l$. Note that due to the Yang–Baxter equation for L-operators, element (4.1) can be rewritten as follows:

$$\mathbb{T}_{[M]}(u_1, \ldots, u_M) = \mathbb{R}^{(M,\ldots,1)}(u_M, \ldots, u_1) \cdot L^{(M)}(u_M) \cdots L^{(1)}(u_1). \tag{4.3}$$

Consider a special multiset $I_{\bar{n}}$ labeled by a sequence of non-negative integers $\bar{n} = \{n_1, n_2, \ldots, n_{N-1}\}$, $\bar{n} \in \mathbb{Z}^{N-1}_{\geq 0}$, $|\bar{n}| = n_1 + \cdots + n_{N-1}$. As an ordered set, $I_{\bar{n}}$ consists of integers i such that $1 \leq i \leq |\bar{n}|$. The colouring map is

$$\iota(i) = a \in \Pi \quad \text{for } 1 + n_1 + \cdots + n_{a-1} \leq i \leq n_1 + \cdots + n_a. \tag{4.4}$$

Let us change the numeration of the set of variables $\{t_i|_{i \in I_{\bar{n}}}\}$ to

$$\bar{t}_{\bar{n}} = \{t^a_i\} = t^1_1, \ldots, t^1_{n_1}, t^2_1, \ldots, t^2_{n_2}, \ldots, t^{N-1}_1, \ldots, t^{N-1}_{n_{N-1}}. \tag{4.5}$$

Following [10], set

$$\begin{aligned}\mathbb{B}_{\bar{n}}(\bar{t}_{\bar{n}}) = &\prod_{a=1}^{N-1} \prod_{1 \leq i < j \leq n_a} \frac{t^a_i - t^a_j}{q^{-1} t^a_i - q t^a_j} \\ &\times ((\mathrm{tr})^{\otimes |\bar{n}|} \otimes \mathrm{id})(\mathbb{T}_{[|\bar{n}|]}(t^1_1, \ldots, t^1_{n_1}; \ldots; t^{N-1}_1, \ldots, t^{N-1}_{n_{N-1}}) \mathsf{e}^{\otimes n_1}_{21} \otimes \cdots \otimes \mathsf{e}^{\otimes n_{N-1}}_{N,N-1} \otimes 1).\end{aligned} \tag{4.6}$$

Here $\mathrm{tr} : \mathrm{End}(\mathbb{C}^N) \to \mathbb{C}$ is the standard trace map. The coefficients of $\mathbb{B}_{\bar{n}}(\bar{t}_{\bar{n}})$ are elements of the Borel subalgebra $U_q(\mathfrak{b}_+)$.

Let $S_{\bar{n}} = S_{n_1} \times \cdots \times S_{n_{N-1}}$ be the direct product of the symmetric groups. The group $S_{\bar{n}}$ naturally acts on functions of $t^1_1, \ldots t^{N-1}_{n_{N-1}}$ by permutations of variables with the same superscript; if $\sigma = \sigma^1 \times \cdots \times \sigma^{N-1} \in S_{\bar{n}}$, then

$${}^{\sigma}\bar{t}_{\bar{n}} = (t^1_{\sigma^1(1)}, \ldots, t^1_{\sigma^1(n_1)}; \ldots; t^{N-1}_{\sigma^{N-1}(1)}, \ldots, t^{N-1}_{\sigma^{N-1}(n_{N-1})}).$$

Proposition 4.1. *For any $\sigma \in S_{\bar{n}}$, we have*

$$\mathbb{B}_{\bar{n}}(\bar{t}_{\bar{n}}) = \mathbb{B}_{\bar{n}}({}^{\sigma}\bar{t}_{\bar{n}}). \tag{4.7}$$

Proof. It suffices to prove the claim assuming that σ is the product of a single simple transposition and the identity permutations. In other words, σ permutes just one pair of variables.

Relations (2.4), the Yang–Baxter equation (2.2) and the inversion relation (2.3) imply that

$$\begin{aligned}&P^{(i,i+1)} R^{(i,i+1)}(u_i, u_{i+1}) \mathbb{T}_{[M]}(u_1, \ldots, u_i, u_{i+1}, \ldots, u_M) \\ &\quad = \mathbb{T}_{[M]}(u_1, \ldots, u_{i+1}, u_i, \ldots, u_M) P^{(i+1,i)} R^{(i+1,i)}(u_{i+1}, u_i),\end{aligned} \tag{4.8}$$

where $P^{(i,i+1)}$ is the permutation operator, $P^{(12)} = \sum_{i,j=1}^{N} \mathsf{e}_{ij} \otimes \mathsf{e}_{ji}$. It is easy to check that

$$P^{(12)} \mathsf{e}_{j+1,j} \otimes \mathsf{e}_{j+1,j} = \mathsf{e}_{j+1,j} \otimes \mathsf{e}_{j+1,j} = \mathsf{e}_{j+1,j} \otimes \mathsf{e}_{j+1,j} P^{(12)} \tag{4.9}$$

and

$$\begin{aligned}R^{(12)}(u_1, u_2) \mathsf{e}_{j+1,j} \otimes \mathsf{e}_{j+1,j} &= \frac{q u_1 - q^{-1} u_2}{u_1 - u_2} \mathsf{e}_{j+1,j} \otimes \mathsf{e}_{j+1,j} \\ &= \mathsf{e}_{j+1,j} \otimes \mathsf{e}_{j+1,j} R^{(12)}(u_1, u_2).\end{aligned} \tag{4.10}$$

If σ permutes just one pair of variables, then relations (4.8)–(4.10) and the cyclic property of the trace yield formula (4.7). For example, in the simplest nontrivial case

$$\begin{aligned}\mathbb{B}(u_2, u_1) &= \frac{u_2 - u_1}{q^{-1} u_2 - q u_1} \mathrm{tr}\left(\mathbb{T}(u_2, u_1) \mathsf{e}_{j+1,j} \otimes \mathsf{e}_{j+1,j}\right) \\ &= \frac{u_2 - u_1}{q^{-1} u_2 - q u_1} \mathrm{tr}\left(P^{(12)} R^{(12)}(u_1, u_2) \mathbb{T}(u_1, u_2) R^{(21)}(u_2, u_1)^{-1} P^{(12)} \mathsf{e}_{j+1,j} \otimes \mathsf{e}_{j+1,j}\right)\end{aligned}$$

$$= \frac{u_1 - u_2}{q^{-1}u_1 - qu_2}\operatorname{tr}\left(\mathbb{T}(u_1, u_2)\mathsf{e}_{j+1,j} \otimes \mathsf{e}_{j+1,j}\right) = \mathbb{B}(u_1, u_2).$$

Proposition 4.1 is proved. □

Let V be a $U_q(\widehat{\mathfrak{gl}}_N)$-module generated by a singular vector v; cf. (2.13). Let $I_{\bar{n}}$ be a special multiset (4.4). Let $\bar{t}_{\bar{n}}$ be the set of variables corresponding to the set $I_{\bar{n}}$. Set

$$\mathbf{w}^{\mathbb{B}}_{V,I_{\bar{n}}}(\{t_i|_{i\in I_{\bar{n}}}\}) = \mathbb{B}_{\bar{n}}(\bar{t}_{\bar{n}})v. \tag{4.11}$$

Clearly, special multisets given by the condition (4.4) are Π-ordered multisets with an increasing colouring function. For a Π-ordered multiset I, set

$$n_a = \#\{i \in I \mid \iota_I(i) = a\},$$

$a = 1, \dots N-1$, and $\bar{n} = (n_1, \dots n_{N-1})$. Let $\sigma : I \to I_{\bar{n}}$ be a unique invertible map intertwining the colouring maps and such that $\sigma(i) \prec \sigma(j)$ iff $\iota_I(i) < i_I(j)$, or $\iota_I(i) = i_I(j)$ and $i \prec j$. Set

$$\mathbf{w}^{\mathbb{B}}_{V,I}(t_i|_{i\in I}) = {}^{\sigma,\tilde{\gamma}}\mathbf{w}^{\mathbb{B}}_{V,I_{\bar{n}}}(t_i|_{i\in I_{\bar{n}}}). \tag{4.12}$$

Theorem 2. *A collection of V-valued rational functions $\mathbf{w}^{\mathbb{B}}_{V,I}(\{t_i|_{i\in I}\})$, given by (4.11), (4.12), is a q-symmetric modified weight function.*

Proof. The collection $\mathbf{w}^{\mathbb{B}}_{V,I}(\{t_i|_{i\in I}\})$ is q-symmetric due to formula (4.12) and Proposition 4.1. Properties (2.16) and (2.17) of the collection $\mathbf{w}^{\mathbb{B}}_{V,I}(\{t_i|_{i\in I}\})$ are straightforward, and the comultiplication property (2.22) follows from Theorem 3.6.3 in [10]. □

5. A correspondence of the two constructions

The goal of this section is to verify the following statement.

Conjecture. *The modified weight functions $\mathbf{w}^{\mathrm{P}}$ and $\mathbf{w}^{\mathbb{B}}$, defined respectively by formulae (3.14) and (4.11), (4.12), coincide.*

The conjecture is equivalent to the following relations. Let v be a weight singular vector in some $U_q(\widehat{\mathfrak{gl}}_N)$-module; cf. (2.13). Take a sequence $\bar{n} = \{n_1, \dots, n_{N-1}\}$, and let $\bar{t}_{\bar{n}}$ be the set of variables (4.5). Then

$$\begin{aligned}\mathbb{B}_{\bar{n}}(\bar{t}_{\bar{n}})v \;&=\; P\left(F_{N-1}(t^{N-1}_{n_{N-1}})\cdots F_{N-1}(t^{N-1}_1)\cdots F_1(t^1_{n_1})\cdots F_1(t^1_1)\right)\, v\\ &\quad\times \prod_{a=1}^{N-1}\left(\prod_{1\le i<j\le n_a}\frac{q^{-1}t^a_i - qt^a_j}{t^a_i - t^a_j}\prod_{1\le i\le n_a}\Lambda_{a+1}(t^a_i)\right).\end{aligned} \tag{5.1}$$

In this paper we will prove the conjecture only for a special case; see Theorem 3. Set

$$R^+(u,v) = \frac{u-v}{qu - q^{-1}v}R(u,v) \in \operatorname{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes \mathbb{C}[[v/u]]$$

and

$$R^-(u,v) = \left(R^+(v,u)^{-1}\right)^{21} \in \operatorname{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes \mathbb{C}[[u/v]],$$

where $R(u,v)$ is defined in (2.1). Define the *evaluation* representation $\pi^{(1)}_z$ of $U_q(\widehat{\mathfrak{gl}}_N)$ in the coordinate space $\mathbb{C}^N$ by the rule $\pi^{(1)}_z\left(L^{\pm}(u)\right) = R^{\pm}(u,z)$. We also denote the representation space of $\pi^{(1)}_z$ as $V_{\omega_1}(z)$. The first coordinate vector in $\mathbb{C}^N$ is a weight singular vector. We denote it by v_{ω_1}.

Theorem 3. *Let V be a subquotient of the tensor product $V_{\omega_1}(z_1)\otimes\cdots\otimes V_{\omega_1}(z_n)$, generated by the singular vector $v=v_{\omega_1}\otimes\cdots\otimes v_{\omega_1}$. Then for any ordered Π-multiset I*

$$\mathbf{w}^{\mathrm{P}}_V(\{t_i|_{i\in I}\})v=\mathbf{w}^{\mathbb{B}}_V(\{t_i|_{i\in I}\})v. \tag{5.2}$$

Proof. Due to the comultiplication properties of the weight functions it is sufficient to prove Theorem 3 for $n=1$. In this case, the weight functions generated by the singular vector v_{ω_1} are nontrivial only if $|I|<N$ and $\iota_I(I)=\{1,2,\ldots,|I|\}$. Since the weight functions $\mathbf{w}^{\mathrm{P}}$ and $\mathbf{w}^{\mathbb{B}}$ are q-symmetric, it is enough to consider the case $I=\{1,2,\ldots,|I|\}$ with the colouring map $\iota_I(i)=i$ for any $i\in I$. In this case, formula (5.2) follows from Proposition 5.1. □

Let the sequence $\bar n$ be such that $n_1=\cdots=n_k=1$, $n_{k+1}=\cdots=n_{N-1}=0$. In this case, the set of variables (4.5) takes the form

$$\bar t=\{t^1,t^2,\ldots,t^k\}, \tag{5.3}$$

where we omit the unnecessary subscript. In addition, write $\mathbb{B}_{[k]}(\bar t)$ instead of $\mathbb{B}_{\bar n}(\bar t)$. In the case described, formula (5.1) reads

$$\mathbb{B}_{[k]}(t^1,\ldots,t^k)\;v=P\left(F_k(t^k)\cdots F_1(t^1)\right)\prod_{j=1}^{k}\Lambda_{j+1}(t^j)v$$

and follows from Proposition 5.1

Say that a vector v is *a singular vector* if

$$E^{+}_{i+1,i}(z)\cdot v=0,\qquad i=1,\ldots,N-1. \tag{5.4}$$

Equivalently, a vector v is a singular vector if

$$L^{+}_{ij}(z)v=0,\qquad 1\le j<i\le N. \tag{5.5}$$

Compared with (2.13), (2.14), here we drop the requirement that the vector v is an eigenvector of the Cartan currents $k^{+}_i(z)$, $i=1,\ldots,N$, and the diagonal entries $L_{ii}(u)$, $i=1,\ldots,N$, of the L-operator. Notice that for a singular vector v, we have

$$k_i(u)\,v=L_{ii}(u)\,v,\qquad i=1,\ldots,N,$$

and $L_{ii}(u)L_{jj}(t)\,v=L_{jj}(t)L_{ii}(u)\,v$ for any i,j.

Proposition 5.1. *Let v be a singular vector. Then for any $k=1,\ldots,N-1$,*

$$\mathbb{B}_{[k]}(t^1,\ldots,t^k)\;v=P\left(F_k(t^k)\cdots F_1(t^1)\right)\,L_{k+1,k+1}(t^k)\cdots L_{22}(t^1)\,v. \tag{5.6}$$

In the rest of this section we are proving Proposition 5.1. The idea of the proof is as follows. We will introduce elements $\mathbb{B}_{[l,k]}(t^l,\ldots,t^k)\in U_q(\mathfrak{b}_+)\subset U_q(\widehat{\mathfrak{gl}}_N)$ such that $\mathbb{B}_{[1,k]}(t^1,\ldots,t^k)=\mathbb{B}_{[k]}(t^1,\ldots,t^k)$ and will obtain relations (5.9) for those elements. We will also consider projections of partial products of currents, $P\left(F_k(t^k)\cdots F_l(t^l)\right)$ and will obtain relations (5.27) for those projections. The fact that relations (5.9) and (5.27) are almost the same will allow us to establish formula (5.6).

5.1. Recurrence relation for $\mathbb{B}_{[l,k]}(t)v$

For any $l=1,\ldots,k$ introduce an element $\mathbb{B}_{[l,k]}(t^l,\ldots,t^k)\in U_q(\mathfrak{b}_+)\subset U_q(\widehat{\mathfrak{gl}}_N)$:

$$\begin{aligned}\mathbb{B}_{[l,k]}(t^l,t^{l+1},\ldots,t^k)={}&\mathrm{tr}_{1,2,\ldots,k-l+1}\Big(\mathbb{R}^{(k-l+1,\ldots,1)}(t^k,\ldots,t^l)\\&\times\;L^{(k-l+1)}(t^k)\cdots L^{(2)}(t^{l+1})L^{(1)}(t^l)\mathrm{e}^{(k-l+1)}_{k+1,k}\cdots\mathrm{e}^{(2)}_{l+2,l+1}\mathrm{e}^{(1)}_{l+1,l}\Big).\end{aligned} \tag{5.7}$$

Recall that for any $A \in \mathrm{End}(\mathbb{C}^N)$ we denote by $A^{(i)} \in \mathrm{End}(\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N)$ the matrix acting as A in the i-th factor of the tensor product $\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N$ and as the identity matrix in all other factors. We also set

$$\mathbb{B}_{[k+1,k]}(\cdot) \equiv 1. \tag{5.8}$$

It is clear that $\mathbb{B}_{[1,k]}(t^1, \ldots, t^k)$ coincides with $\mathbb{B}_{[k]}(t^1, \ldots, t^k)$.

Let v be a singular weight vector. Recall that $L_{i,i}(t)\, v = k_i(t)\, v = \Lambda_i(t)\, v$. We will show that the action of the element $\mathbb{B}_{[l,k]}(t^l, \ldots, t^k)$ on the singular vector v can be expressed using a linear combination of products of the Gauss coordinates $F^+_{i,j}(t^{j-1})$ with $l \le i < j \le k+1$. For example, we have $\mathbb{B}_{[k,k]}(t^k) = L_{k,k+1}(t^k)$, so that $\mathbb{B}_{[k,k]}(t^k)\, v = F^+_{k,k+1}(t^k) L_{k+1,k+1}(t^k)\, v$, and

$$\mathbb{B}_{[k-1,k]}(t^{k-1}, t^k) = L_{k,k+1}(t^k) L_{k-1,k}(t^{k-1}) + \frac{(q-q^{-1})\, t^k}{t^k - t^{k-1}}\, L_{k-1,k+1}(t^k) L_{k,k}(t^{k-1})$$

so that

$$\mathbb{B}_{[k-1,k]}(t^{k-1}, t^k)\, v = \left(F^+_{k,k+1}(t^k)\, F^+_{k-1,k}(t^{k-1}) + \frac{(q-q^{-1})\, t^k}{t^k - t^{k-1}}\, F^+_{k-1,k+1}(t^k) \right) L_{k+1,k+1}(t^k)\, L_{k,k}(t^{k-1})\, v.$$

To obtain the required presentation in general, we will use the following statement.

Proposition 5.2. *We have*

$$\mathbb{B}_{[l,k]}(t^l, \ldots, t^k)\, v = \sum_{m=l+1}^{k+1} \mathbb{B}_{[m,k]}(t^m, \ldots, t^k) F^+_{l,m}(t^{m-1}) \times L_{m,m}(t^{m-1}) \cdots L_{l+1,l+1}(t^l)\, v \cdot \prod_{j=l+1}^{m-1} \frac{(q-q^{-1})\, t^j}{t^j - t^{j-1}}. \tag{5.9}$$

We start the proof of this proposition from the next lemma.

Lemma 5.3.

$$\begin{aligned} \mathbb{B}_{[l,k]}(t^l, \ldots, t^k)\, v = {} & \mathbb{B}_{[l+1,k]}(t^{l+1}, \ldots, t^k) \cdot L_{l,l+1}(t^l)\, v \\ & + \mathrm{tr}_{2,\ldots,k-l+1} \left(\mathbb{R}^{(k-l+1,\ldots,2)}(t^k, \ldots, t^{l+1}) L^{(k-l+1)}(t^k) \cdots L^{(2)}(t^{l+1}) \right. \\ & \left. \times\ \mathsf{e}^{(k-l+1)}_{k+1,k} \cdots \mathsf{e}^{(3)}_{l+3,l+2} \mathsf{e}^{(2)}_{l+2,l} \right) \frac{(q-q^{-1})\, t^{l+1}}{t^{l+1} - t^l} L_{l+1,l+1}(t^l)\, v. \end{aligned} \tag{5.10}$$

Proof. To obtain formula (5.10) we calculate the trace over the first copy of $\mathbb{C}^N$ in formula (5.7). Using the Yang–Baxter equation (2.2), we get

$$\mathbb{R}^{(k-l+1,\ldots,2,1)}(t^k, \ldots, t^{l+1}, t^l) = R^{(2,1)}(t^{l+1}, t^l) \cdots R^{(k-l+1,1)}(t^{k-l+1}, t^l) \mathbb{R}^{(k-l+1,\ldots,2)}(t^k, \ldots, t^{l+1}).$$

Due to relations (2.14), we can write the right hand side of formula (5.7) applied to the weight singular vector v as a sum of two terms:

$$\begin{aligned} & \mathrm{tr}_{2,\ldots,k-l+1} \left(\mathrm{tr}_1 \left(R^{(21)}(t^{l+1}, t^l) \cdots R^{(k-l+1,1)}(t^k, t^l)\, \mathsf{e}^{(1)}_{ll} \right) \cdot X \cdot L_{l,l+1}(t^l)\, v \right. \\ & \quad \left. + \mathrm{tr}_1 \left(R^{(21)}(t^{l+1}, t^l) \cdots R^{(k-l+1,1)}(t^k, t^l)\, \mathsf{e}^{(1)}_{l+1,l} \right) \cdot X \cdot L_{l+1,l+1}(t^l)\, v \right), \end{aligned} \tag{5.11}$$

where

$$X = \mathbb{R}^{(k-l+1,\ldots,2)}(t^k, \ldots, t^{l+1})\, L^{(k-l+1)}(t^k) \cdots L^{(2)}(t^{l+1})\, \mathsf{e}^{(k-l+1)}_{k+1,k} \cdots \mathsf{e}^{(2)}_{l+2,l+1}.$$

Now we calculate the traces tr_1 in formula (5.11) taking into account the matrix structure of the R-matrix (2.1) and the multiplication rule: $\mathsf{e}_{ab}\,\mathsf{e}_{cd}=0$ for $b\neq c$ and $\mathsf{e}_{ab}\,\mathsf{e}_{bc}=\mathsf{e}_{ac}$. As a result, we get

$$\mathrm{tr}_1\left(R^{(21)}(t^{l+1},t^l)\cdots R^{(k-l+1,1)}(t^k,t^l)\,\mathsf{e}^{(1)}_{ll}\right)=\mathbf{1}^{(2)}\,\mathbf{1}^{(3)}\cdots\mathbf{1}^{(k-l+1)}+\sum_{i=2}^{k-l+1}\mathsf{e}^{(i)}_{ll}\,Y_i \tag{5.12}$$

and

$$\mathrm{tr}_1\left(R^{(21)}(t^{l+1},t^l)\cdots R^{(k-l+1,1)}(t^k,t^l)\,\mathsf{e}^{(1)}_{l+1,l}\right)$$
$$=\frac{(q-q^{-1})\,t^{l+1}}{t^{l+1}-t^l}\mathsf{e}^{(2)}_{l+1,l}\,\mathbf{1}^{(3)}\cdots\mathbf{1}^{(k-l+1)}+\sum_{i=3}^{k-l+1}\mathsf{e}^{(i)}_{l+1,l}\,Y'_i, \tag{5.13}$$

where Y_i, Y'_i are some elements of $\mathrm{End}(\mathbb{C}^N\otimes\cdots\otimes\mathbb{C}^N)$. Observe that only the first terms in the right hand sides of formulae (5.12) and (5.13) contribute nontrivially to the trace $\mathrm{tr}_{2,\ldots,k-l+1}$ in formula (5.11) because $\mathrm{tr}\,(\mathsf{e}_{ab}\,A\,\mathsf{e}_{cd})=0$ for any $A\in\mathrm{End}(\mathbb{C}^N)$ unless $a=d$. To complete the proof of Lemma 5.3, notice that

$$\mathrm{tr}_{2,\ldots,k-l+1}\,X=\mathbb{B}_{[l+1,k]}(t^{l+1},\ldots,t^k)$$

and $\mathrm{tr}\,(\mathsf{e}_{l+1,l}\,A\,\mathsf{e}_{l+2,l+1})=\mathrm{tr}\,(A\,\mathsf{e}_{l+2,l})$ for any $A\in\mathrm{End}(\mathbb{C}^N)$. □

Proof of Proposition 5.2. To prove this proposition we use induction with respect to N.

The first term in the right hand side of formula (5.10) is exactly the term in the right hand side of formula (5.9) for $m=l+1$, because $L_{l,l+1}(t^l)\,v=F^+_{l,l+1}(t^l)\,L_{l+1,l+1}(t^l)\,v$, and it suffices to show that

$$\mathrm{tr}_{2,\ldots,k-l+1}(\mathbb{R}^{(k-l+1,\ldots,2)}(t^k,\ldots,t^{l+1})L^{(k-l+1)}(t^k)\cdots L^{(2)}(t^{l+1})\mathsf{e}^{(k-l+1)}_{k+1,k}\cdots\mathsf{e}^{(3)}_{l+3,l+2}\mathsf{e}^{(2)}_{l+2,l})\,v$$
$$=\sum_{m=l+2}^{k+1}\mathbb{B}_{[m,k]}(t^m,\ldots,t^k)F^+_{l,m}(t^{m-1})\,L_{mm}(t^{m-1})\cdots L_{l+2,l+2}(t^{l+1})\,v\cdot\prod_{j=l+2}^{m-1}\frac{(q-q^{-1})\,t^j}{t^j-t^{j-1}}. \tag{5.14}$$

Consider the embedding $\psi: U_q(\widehat{\mathfrak{gl}}_{N-1})\hookrightarrow U_q(\widehat{\mathfrak{gl}}_N)$ given by the rule

$$\psi\left(L^{[N-1]}_{ij}(t)\right)=L_{i+\theta(i>l),j+\theta(j>l)}(t),\quad i,j=1,\ldots,N-1,$$

where $\theta(m>l)=0$ for $m\le l$, and $\theta(m>l)=1$ for $m>l$. Assume that $U_q(\widehat{\mathfrak{gl}}_{N-1})$ acts by the composition of the embedding ψ and the action of $U_q(\widehat{\mathfrak{gl}}_N)$. Then the vector v is singular with respect to the action of $U_q(\widehat{\mathfrak{gl}}_{N-1})$. Taking into account the matrix structure of the R-matrix (2.1), we can verify that $\psi\left(\mathbb{B}^{[N-1]}_{[m-1,k-1]}(t^m,\ldots,t^k)\right)=\mathbb{B}_{[m,k]}(t^m,\ldots,t^k)$ for $m>l$, and the left hand side of formula (5.14) coincides with $\psi\left(\mathbb{B}^{[N-1]}_{[l,k-1]}(t^{l+1},\ldots,t^k)\right)$. In addition, observe that

$$\psi\left(\left(F^+_{l,m-1}\right)^{[N-1]}(t)\right)=F^+_{l,m}(t),\quad m\ge l+2.$$

As a result, taking formula (5.9) for $U_q(\widehat{\mathfrak{gl}}_{N-1})$ with parameters $l,k-1,t^{l+1},\ldots t^k$, and applying the embedding ψ we obtain formula (5.14). Proposition 5.2 is proved. □

5.2. *Composed currents and Gauss coordinates*

In the next two subsections we will show that the projections of products of currents, $P\left(F_k(t^k)\cdots F_l(t^l)\right)$, satisfy relations (5.27) which are similar to relations (5.9) for the elements $\mathbb{B}_{[l,k]}(t^l,\ldots,t^k)$. We will use those relations in Section 5.4 to prove Proposition 5.1.

Following [3,4,7], we will introduce the *composed currents* $F_{i,j}(t)$ for $i<j$; see (5.15). The composed currents for an arbitrary quantum affine algebra were defined in [3]. The currents $F_{i,j}(t)$ to be used here are images of the composed currents for the algebra $U_q(\widehat{\mathfrak{sl}}_N)$ under the embedding $\Theta: U_q(\widehat{\mathfrak{sl}}_N)\hookrightarrow U_q(\widehat{\mathfrak{gl}}_N)$ defined in Section 3.2.

The currents $F_{i,i+1}(t)$, $i=1,\dots N-1$, are just the currents $F_i(t)$; cf. (2.6). It follows from formulae (2.10) and the definition of the projection P, see (3.10), that $P\left(F_{i,i+1}(t)\right)=F^+_{i,i+1}(t)$, that is, the projection of the current $F_{i,i+1}(t)$ coincides with the Gauss coordinate $F^+_{i,i+1}(t)$ of the corresponding L-operator [2]. There exists a similar relation between other Gauss coordinates $F^+_{i,j}(t)$ and projections of the composed currents $F_{i,j}(t)$; see Proposition 5.5.

According to [3], the composed currents $F_{i,j}(t)$ belong to a suitable completion of the $U_q(\widehat{\mathfrak{gl}}_N)$ subalgebra generated by modes $F_i[n]$, $n\in\mathbb{Z}$, $i=1,\dots,N-1$. Elements of the completion are infinite sums of monomials which are ordered products $F_{i_1}[n_1]\cdots F_{i_k}[n_k]$ with $n_1\le\cdots\le n_k$. We denote this completion by $\overline{U}_f$.

The completion $\overline{U}_f$ determines analyticity properties of products of currents; see [3]. One can show that for $|i-j|>1$, the product $F_i(t)F_j(w)$ is an expansion of a function analytic at $t\neq 0$, $w\neq 0$. The situation is more delicate for $j=i,i\pm 1$. The products $F_i(t)F_i(w)$ and $F_i(t)F_{i+1}(w)$ are expansions of analytic functions at $|w|<|q^2t|$, while the product $F_i(t)F_{i-1}(w)$ is an expansion of an analytic function at $|w|<|t|$. Moreover, the only singularities of the corresponding functions in the whole region $t\neq 0$, $w\neq 0$, are simple poles at the respective hyperplanes, $w=q^2t$ for $j=i,i+1$, and $w=t$ for $j=i-1$.

The composed currents $F_{i,j}(t)$, $i<j$, are given by the rule

$$F_{i,j}(t)=(q-q^{-1})^{j-i-1}F_i(t)F_{i+1}(t)\cdots F_{j-1}(t). \tag{5.15}$$

For example, $F_{i,i+1}(t)=F_i(t)$, and $F_{i,i+2}(t)=(q-q^{-1})F_i(t)F_{i+1}(t)$. The last product is well defined according to the analyticity properties of the product $F_i(t)F_{i+1}(w)$, described above. In a similar way, one can show inductively that the product in the right hand side of (5.15) makes sense for any $i<j$.

Products of the composed currents have the following analyticity properties. For any $i<r<s<j$, the products $F_{i,r}(t)F_{s,j}(w)$ and $F_{s,j}(t)F_{i,r}(w)$ are expansions of functions analytic at $t\neq 0$, $w\neq 0$. For any $i<s<j$, the product $F_{i,s}(t)F_{s,j}(w)$ is an expansion of an analytic function at $|w|<|q^2t|$, and the product $F_{s,j}(t)F_{i,s}(w)$ is an expansion of an analytic function at $|w|<|t|$. Moreover, the only singularities of the corresponding functions in the whole region $t\neq 0$, $w\neq 0$ are simple poles at the respective hyperplanes, $w=q^2t$ for $F_{i,s}(t)F_{s,j}(w)$, and $w=t$ for $F_{s,j}(t)F_{i,s}(w)$.

The composed currents obey commutation relations

$$(q^{-1}w-qt)F_{i,s}(w)F_{s,j}(t)=(w-t)F_{s,j}(t)F_{i,s}(w), \tag{5.16}$$

for any $i<s<j$, and

$$F_{i,r}(w)F_{s,j}(t)=F_{s,j}(t)F_{i,r}(w), \tag{5.17}$$

for any $i<r<s<j$, which can be observed from the basic relations (2.7) and formula (5.15). In addition, the residue formula

$$F_{i,j}(t)=-\operatorname*{res}_{w=t}F_{s,j}(t)F_{i,s}(w)\,\frac{dw}{w} \tag{5.18}$$

holds for any $s=i+1,\dots,j-1$. Since the total sum of residues of an analytic functions equals zero, taking into account commutation relations (5.16), we also get

$$F_{i,j}(t)=\operatorname*{res}_{w=0}\left(F_{s,j}(t)F_{i,s}(w)\frac{dw}{w}\right)+\operatorname*{res}_{w=\infty}\left(\frac{q^{-1}w-qt}{w-t}\,F_{i,s}(w)F_{s,j}(t)\frac{dw}{w}\right). \tag{5.19}$$

Set $S_A(B)=BA-qAB$. Projections of composed currents can be defined using q-commutators with zero modes of the currents $F_i(t)$, $i=1,\dots,N-1$. We will call operators $S_{F_i[0]}$ *the screening operators*.

Proposition 5.4. *We have*

$$P\left(F_{i,j}(t)\right)=S_{F_i[0]}\left(P(F_{i+1,j}(t))\right),\quad i<j-1. \tag{5.20}$$

Proof. Calculating the residues in the right hand side of formula (5.19) for $s = i+1$ and using the fact that $F_{i,i+1}(t) = F_i(t)$, we obtain

$$F_{i,j}(t) = F_{i+1,j}(t)F_i[0] - qF_i[0]F_{i+1,j}(t) + (q-q^{-1})\sum_{k\le 0} F_i[k]\, F_{i+1,j}(t)\, t^{-k}. \tag{5.21}$$

Now we apply the projection P, see (3.10), to both sides of this relation. The modes $F_i[k]$ with $k \le 0$ belong to U_f^-. Hence, due to formulae (3.10), the projection P kills the semi-infinite sum in the right hand side of (5.21), and we get

$$\begin{aligned} P\left(F_{i,j}(t)\right) &= P\left(F_{i+1,j}(t)F_i[0] - q\, F_i[0]F_{i+1,j}(t)\right) \\ &= P\left(S_{F_i[0]}\left(F_{i+1,j}(t)\right)\right) = S_{F_i[0]}\left(P\left(F_{i+1,j}(t)\right)\right). \end{aligned} \tag{5.22}$$

To get the last equality we use the fact proved in [4,7] that the projection P commutes with the screening operators $S_{F_i[0]}$, $P\left(S_{F_i[0]}(F)\right) = S_{F_i[0]}\left(P(F)\right)$ for any $F \in U_F'$. □

Proposition 5.5. *We have*

$$P\left(F_{i,j}(t)\right) = (q-q^{-1})^{j-i-1} F^+_{i,j}(t), \quad i < j-1. \tag{5.23}$$

Proof. The claim follows by induction with respect to $j-i$ from formula $P\left(F_{i,i+1}(u)\right) = F^+_{i,i+1}(u)$, Proposition 5.4 and Lemma 5.6 proved below. □

Lemma 5.6. *We have*

$$(q-q^{-1})F^+_{i,j}(t) = S_{F_i[0]}\left(F^+_{i+1,j}(t)\right), \quad i < j-1. \tag{5.24}$$

Proof. It follows from relation (2.9) that

$$L^+_{i,j}(t) = F^+_{i,j}(t)k^+_j(t) + \sum_{m=j+1}^{N} F^+_{i,m}(t)k^+_m(t)E^+_{m,j}(t). \tag{5.25}$$

Since $S_{F_i[0]}\left(F^+_{i+1,j}(t)\right) = F^+_{i+1,j}(t)F_i[0] - qF_i[0]F^+_{i+1,j}(t)$, and taking into account the commutativity

$$[F_i[0], k^+_j(t)] = 0 \quad \text{and} \quad [F_i[0], k^+_m(t)E^+_{m,j}(t)] = 0$$

for $i = 1, \ldots, j-2$ and $m = j+1, \ldots, N$, we observe that relation (5.24) results from formula (5.25) and the equality

$$(q-q^{-1})L^+_{i,j}(t) = L^+_{i+1,j}(t)F_i[0] - qF_i[0]L^+_{i+1,j}(t) \tag{5.26}$$

by induction with respect to j starting from $j = N$. On the other hand, the second line in (2.4) at $w = 0$ gives

$$L^+_{i+1,j}(t)L^-_{i,i+1}[0] + (q-q^{-1})L^+_{i,j}(t)L^-_{i+1,i+1}[0] = L^-_{i,i+1}[0]L^+_{i+1,j}(t),$$

which yields formula (5.26), if we keep in mind the relations

$$\begin{aligned} &L^-_{i,i+1}(0) = L^-_{i,i+1}[0] = -F_i[0]k^{-1}_{i+1}, \qquad L^-_{i,i}(0) = L^-_{i,i}[0] = k_i^{-1}, \\ &k^{-1}_{i+1}L^+_{i+1,j}(t)k_{i+1} = qL^+_{i+1,j}(t), \end{aligned}$$

also following from (2.4), (2.9). □

5.3. Calculation of the projections

The following proposition is a counterpart of Proposition 5.2.

Proposition 5.7. *For any $k > l$, we have*

$$P\left(F_{k,k+1}(t^k)\cdots F_{l,l+1}(t^l)\right) = \sum_{m=l+1}^{k+1} P\left(F_{k,k+1}(t^k)\cdots F_{m,m+1}(t^m)\right) P\left(F_{l,m}(t^{m-1})\right) \prod_{j=l+1}^{m-1} \frac{t^j}{t^j - t^{j-1}}. \tag{5.27}$$

Proof. The claim follows from Lemma 5.8 proved below. □

Lemma 5.8. *For any $j = l+1, \ldots, k$, we have*

$$\begin{aligned} &P\left(F_{k,k+1}(t^k)F_{k-1,k}(t^{k-1})\cdots F_{j+1,j+2}(t^{j+1})F_{j,j+1}(t^j)F_{l,j}(t^{j-1})\right)\\ &\quad= P\left(F_{k,k+1}(t^k)F_{k-1,k}(t^{k-1})\cdots F_{j+1,j+2}(t^{j+1})F_{j,j+1}(t^j)\right) P\left(F_{l,j}(t^{j-1})\right)\\ &\qquad+ \frac{t^j}{t^j - t^{j-1}} P\left(F_{k,k+1}(t^k)F_{k-1,k}(t^{k-1})\cdots F_{j+1,j+2}(t^{j+1})F_{l,j+1}(t^j)\right). \end{aligned} \tag{5.28}$$

Proof. We will prove the lemma by induction with respect to l, decreasing l from $j-1$ to 1.

We will use the following properties of the projection P coming from the definition (3.10):

$$P(F^-_{s,s+1}(t)\cdot X) = 0, \tag{5.29}$$

for any $s = 1, \ldots N-1$, and $X \in U_f$, and

$$P(X_1 P(X_2)) = P(X_1)P(X_2), \tag{5.30}$$

for any $X_1, X_2 \in U_f$.

Consider the case $l = j-1$. To calculate the left hand side of formula (5.28) we split the current $F_{l,j}(t^{j-1}) = F_{j-1,j}(t^{j-1})$:

$$F_{j-1,j}(t^{j-1}) = F^+_{j-1,j}(t^{j-1}) - F^-_{j-1,j}(t^{j-1}).$$

Since $F^+_{j-1,j}(t^{j-1}) = P\left(F_{j-1,j}(t^{j-1})\right)$, the first term here produces the first term in the right hand side of formula (5.28). For the second term, we move the negative half-current $F^-_{j-1,j}(t^{j-1})$ to the left using the relation

$$\begin{aligned} F_{j,j+1}(t^j)F^-_{j-1,j}(t^{j-1}) &= \frac{qt^j - q^{-1}t^{j-1}}{t^j - t^{j-1}}\, F^-_{j-1,j}(t^{j-1})F_{j,j+1}(t^j)\\ &\quad - \frac{(q-q^{-1})\,t^j}{t^j - t^{j-1}}\, F^-_{j-1,j}(t^j)F_{j,j+1}(t^j) + \frac{t^j}{t^j - t^{j-1}}\, F_{j-1,j+1}(t^j), \end{aligned} \tag{5.31}$$

which is a consequence of formulae (5.16), (5.18) and the analyticity properties of the products of currents, and the fact that the currents $F_{s,s+1}(t^s)$ for $s > j$ commute with $F^-_{j-1,j}(t^{j-1})$. Due to relation (5.29), only the third term in the right hand side of (5.31) contributes nontrivially to the projection, and we obtain the second term in the right hand side of formula (5.28) for $l = j-1$,

$$\begin{aligned} &P\left(F_{k,k+1}(t^k)\cdots F_{l+1,l+2}(t^{l+1})\right)\cdot P\left(F_{l,l+1}(t^l)\right)\\ &\quad+ P\left(F_{k,k+1}(t^k)\cdots F_{l+2,l+3}(t^{l+2})F_{l,l+2}(t^{l+1})\right)\frac{t^{l+1}}{t^{l+1} - t^l}. \end{aligned} \tag{5.32}$$

Assume now that $l \le j-2$. Formula (5.21) gives that

$$F_{l,s}(t) = S_{F_l[0]}\left(F_{l+1,s}(t)\right) - (q-q^{-1})F^-_{l,l+1}(t)\, F_{l+1,s}(t). \tag{5.33}$$

We replace $F_{l,j}(t^{j-1})$ and $F_{l,j+1}(t^j)$ in (5.28) by the right hand side of formula (5.33) for $s = j, j+1$, respectively. Since the currents $F_{s,s+1}(t^s)$ for $s > l+1$ commute with $F^-_{l,l+1}(t)$, the contributions of the second term in (5.33) vanish due to relation (5.29). For the first term, we use the fact that $P\left(S_{F_l[0]}(F)\right) = S_{F_l[0]}\left(P(F)\right)$ for any $F \in U'_F$, see [4, 7], and the commutativity of the currents $F_{s,s+1}(t^s)$ for $s > l+1$ with $F_l[0]$. As a result, we get that formula (5.28) is equivalent to

$$\begin{aligned} &S_{F_l[0]}\left(P\left(F_{k,k+1}(t^k)F_{k-1,k}(t^{k-1})\cdots F_{j+1,j+2}(t^{j+1})F_{j,j+1}(t^j)F_{l+1,j}(t^{j-1})\right)\right)\\ &\quad = S_{F_l[0]}\left(P\left(F_{k,k+1}(t^k)F_{k-1,k}(t^{k-1})\cdots F_{j+1,j+2}(t^{j+1})F_{j,j+1}(t^j)\right)P\left(F_{l+1,j}(t^{j-1})\right)\right)\\ &\quad\quad + \frac{t^j}{t^j - t^{j-1}} S_{F_l[0]}\left(P\left(F_{k,k+1}(t^k)F_{k-1,k}(t^{k-1})\cdots F_{j+1,j+2}(t^{j+1})F_{l+1,j+1}(t^j)\right)\right). \end{aligned}$$

The last equality is obtained by application of the screening operator $S_{F_l[0]}$ to formula (5.28) with l replaced by $l+1$, and is true by the induction assumption. □

5.4. *Proof of Proposition 5.1*

For each $l = 1, \ldots N-2$ we consider the embedding $\psi_l : U_q(\widehat{\mathfrak{gl}}_{N-l}) \hookrightarrow U_q(\widehat{\mathfrak{gl}}_N)$, defined by the rule

$$\psi_l\left(L^{[N-l]}_{ij}(t)\right) = L_{i+l,\,j+l}(t), \quad i, j = 1, \ldots, N-l.$$

Taking into account the matrix structure of the R-matrix (2.1), one can verify that for $l < m < k$,

$$\psi_l\left(\mathbb{B}^{[N-l]}_{[m-l,k-l]}(t^m, \ldots, t^k)\right) = \mathbb{B}_{[m,k]}(t^m, \ldots, t^k).$$

In addition, using formula (2.9) one can check that for $l < i$, $\psi_l\left(F^{[N-l]}_{i-l}(t)\right) = F_i(t)$. Besides this, the embedding ψ_l is consistent with the projections $P^{[N-l]}$ and P.

We prove Proposition 5.1 by induction with respect to N. We replace the expressions in both sides of formula (5.6) by the right hand sides of formulae (5.9) and (5.27) with $l = 1$, respectively, and compare the results term by term. The terms for $m = k+1$ are manifestly the same, taking into account formula (5.23). For $m = 2, \ldots k$, the equality of the corresponding terms is equivalent to

$$\begin{aligned} &\psi_{m-1}\left(\mathbb{B}^{[N-m+1]}_{[k-m+1]}(t^m, \ldots, t^k)\right) F^+_{1,m}(t^{m-1})\, L_{mm}(t^{m-1}) \cdots L_{22}(t^1)\, v\\ &\quad = \psi_{m-1}\left(P^{[N-m+1]}(F^{[N-m+1]}_{k-m+1}(t^k) \cdots F^{[N-m+1]}_1(t^m))\right)\\ &\quad\quad \times F^+_{1,m}(t^{m-1})\, L_{k+1,k+1}(t^k) \cdots L_{22}(t^1)\, v. \end{aligned} \tag{5.34}$$

It follows from commutation relations (2.4), (2.7), that if v is a singular vector with respect to the action of $U_q(\widehat{\mathfrak{gl}}_N)$, then the vector

$$v_{m-1} = F^+_{1,m}(t^{m-1})\, L_{mm}(t^{m-1}) \cdots L_{22}(t^1)\, v$$

is a singular vector with respect to the action of $U_q(\widehat{\mathfrak{gl}}_{N-m+1})$ induced by the embedding $\psi_{m-1} : U_q(\widehat{\mathfrak{gl}}_{N-m+1}) \hookrightarrow U_q(\widehat{\mathfrak{gl}}_N)$, and

$$F^+_{1,m}(t^{m-1})\, L_{k+1,k+1}(t^k) \cdots L_{22}(t^1)\, v = L_{k+1,k+1}(t^k) \cdots L_{m+1,m+1}(t^m)\, v_{m-1}.$$

Hence, formula (5.34) takes the form

$$\begin{aligned} &\psi_{m-1}\left(\mathbb{B}^{[N-m+1]}_{[k-m+1]}(t^m, \ldots, t^k)\right) v_{m-1}\\ &\quad = \psi_{m-1}(P^{[N-m+1]}(F^{[N-m+1]}_{k-m+1}(t^k) \cdots F^{[N-m+1]}_1(t^m)) L_{k-m+2,k-m+2}(t^k) \cdots L_{22}(t^m))\, v_{m-1}, \end{aligned}$$

which follows from the induction assumption. □

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